Physics IV
ISI B.Math
Final Exam : May 4, 2012

Total Marks: 80
Time : 3 hours
Answer all questions

1. (**Marks : 6 + 4 + 5 + 5 = 20**)

A particle of mass m is confined to a one dimensional region $0 \leq x \leq a$ by the potential $V(x) = 0$ for $0 \leq x \leq a$ and $V(x) = \infty$ for $x > a, x < a$.

(a) Find the energy eigenvalues and corresponding eigenfunctions for this system.

At $t = 0$, it is given that the normalized wave function is

$$\psi(x, t = 0) = \sqrt{\frac{8}{5a}} \left[1 + \cos\left(\frac{\pi x}{a}\right)\right] \sin\left(\frac{\pi x}{a}\right)$$

(b) What is the probability that the particle is found in the left half of the box (i.e, in the region $0 \leq x \leq \frac{a}{2}$) at $t = 0$?

(c) What is the expectation value of the energy of the system at $t = 0$?

(d) What is the wave function at a later time t ?

2. (**Marks: 4 + 4 + 4 + 8 = 20**)

The annihilation operator for the harmonic oscillator in 1-d is defined as $\hat{a} = \sqrt{\frac{m\omega_0}{2\hbar}} \left(\hat{x} + i\frac{\hat{p}}{m\omega_0}\right)$ where m is the mass and ω_0 is the angular frequency of the harmonic oscillator

(a) If $\hat{H}$ is the Hamiltonian and $\hat{N} = \hat{a}^\dagger \hat{a}$ is the number operator, show that $\hat{N}$ and $\hat{H}$ have simultaneous eigenstates

(b) Show that $< \hat{H} > \geq 0$

(c) Given that $|n>$ is a normalized eigenstate corresponding to the eigenvalue n of the Hamiltonian of the harmonic oscillator where n is an integer and $\hat{a}|n> = \sqrt{n}|n-1>$ and $\hat{a}^\dagger|n> = \sqrt{n+1}|n+1>$, show that in the nth energy eigenstate of the harmonic oscillator, $< \hat{x} > = < \hat{p} > = 0$

(d) Show that the average kinetic energy $< T >$ is equal to the average potential energy $< V >$ for any energy eigenstate of the harmonic oscillator

3. (**Marks: 12**)
For a certain system, the operator corresponding to the physical quantity A does not commute with the Hamiltonian. It has eigenvalues α_1 and α_2 corresponding to properly normalized eigenfunctions

$$\phi_1 = \frac{(u_1 + u_2)}{\sqrt{2}}$$

$$\phi_2 = \frac{(u_1 - u_2)}{\sqrt{2}}$$

where u_1 and u_2 are properly normalized eigenfunctions of the Hamiltonian with eigenvalues E_1 and E_2. If the system is in the state $\psi = \phi_1$ at time $t = 0$, show that the expectation value of A at time t is

$$< A >= \left(\frac{\alpha_1 + \alpha_2}{2}\right) + \left(\frac{\alpha_1 - \alpha_2}{2}\right) \cos\left(\frac{|E_1 - E_2|t}{\hbar}\right)$$

4. (**Marks: 10**)

The wave function of a free particle at $t = 0$ is given by

$$\psi(x, 0) = \frac{1}{\sqrt{a}} \text{ for } -\frac{a}{2} \leq x \leq \frac{a}{2}$$

$$\text{and } \psi(\mathrm{x}, 0) = 0 \text{ elsewhere.}$$

If the momentum of the particle is measured at $t = 0$, show that the probability $P(k)$ of finding the momentum of the particle between $\hbar k$ and $\hbar(k + dk)$ is given by

$$P(k) = \frac{2}{\pi a}\frac{\sin^2(ka/2)}{k^2}$$

What is the most probable value of the momentum that will be found on measurement? Which values of momentum will never be found ?
(Hint: Recall that the eigenstates of momentum $\hat{p}$ with eigenvalue $\hbar k$ are given by $\frac{1}{\sqrt{2\pi}}e^{ikx}$ and expand the wavefunction as a superposition of these eigenstates)

5. (**Marks: 3 + 3 + 2 + 5 + 5 = 18**)

(a) A stick of length L moves past you at speed v. There is a time interval between the front end of the stick coinciding with you and the back end coinciding with you. What is this time interval in
(i) your frame? (Calculate this by working in your frame)
(ii)your frame?(work in the stick's frame)
(iii)the stick's frame? (work in the stick's frame)
(iv) the stick's frame? (work in your frame)

(b) Show that it is impossible for an isolated free electron to emit or absorb a photon.
(Hint: Use the conservation of four momentum)